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# On the minimal relative motion principle—the oscillating tilted bar

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**Abstract**

Beghi, Xausa, Tomat, and Zanforlin (J. Math. Psychol. 41(1997) 11) present a visual stereokinetic illusion. In the image plane, one end of an oblique bar moves horizontally back and forth, while the other end is stationary. Perceptually, this becomes a bar of a constant length rotating in depth around a vertical axis that passes through the stationary end of the bar. Beghi et al. (1997) provide a mathematical model of minimal relative motion to account for this percept. Here we show that the minimal relative motion principle cannot explain the perceptual phenomenon. Specifically, we raise two objections. (1) It is necessary to consider not only the length, but also the direction, of a vector when comparing vector fields. In fact, when directions are taken into consideration, Beghi et al.'s mathematical result diverges from their perceptual experimental result. (2) There is a mathematical inconsistency in Beghi et al. (1997): mixing absolute and relative velocities in their minimization is unwarranted, and does not lead to correct minimization.


*Keywords:* Stereokinetic; Rigidity; Structure from motion; Depth; Minimal relative motion

## 1. Introduction

We will first introduce the original model by Beghi, Xausa, Tomat, and Zanforlin (1997) and, for clarity, we will use the same notations. We will then present our two-point critique. As shown in Fig. 1, bar $OP'$ is in the image plane $Oxz$. Its end point $O$ remains stationary, while the other end $P'$ moves horizontally back and forth with a constant speed $v$. This stimulus is perceived, according to Beghi et al. (1997), as a bar of a constant length oscillating in depth around axis $Oz$. In order to account for this percept, Beghi et al. (1997) propose a minimal relative motion principle that minimizes speed differences between all points on the bar. To quote Beghi et al. (1997, p. 12):

> "[I]f the points of a pattern moving on the frontal plane have different velocities, the differences between the lengths of these velocity vectors can be annulled or minimized by adding to them a velocity component oriented in depth, in such a way that the length of these new velocity vectors is equal for all the points… The velocities to be taken into consideration… are [the] relative velocities with respect to the perceptual centre of [the] image."

The perceptual center, $M'$ (the midpoint of the bar), and point $P'$ of the bar, have the following velocity components in the image plane $Oxz$:

$$v_{P'_x} = v, \qquad v_{M'_x} = \tfrac{1}{2}v, \qquad v_{P'_z} = v_{M'_z} = 0. \tag{1}$$

Let $Q'$ be an arbitrary point on the bar $OP'$ such that $OQ' = \lambda OP', 0 \leqslant \lambda \leqslant 1$, then

$$v_{Q'_x} = \lambda v, \qquad v_{Q'_z} = 0. \tag{2}$$

Therefore, the velocity of $Q'$ relative to the midpoint $M'$ is (Eqs. (2) and (3) of Beghi et al. (1997, p. 12)):

$$v^*_{Q'_x} = v_{Q'_x} - v_{M'_x} = v\left(\lambda - \tfrac{1}{2}\right), \qquad v^*_{Q'_z} = 0. \tag{3}$$

The next step is crucial, where Beghi et al. (1997) apply the minimal relative motion principle (Eq. (4) of Beghi et al. (1997, p. 13)): A velocity component in depth $v_{Q'_y}$ is added, where the $y$-axis is orthogonal to the image plane and pointing in the viewing direction. This new component in depth is chosen such that

$$\left(v^*_{Q'_x}\right)^2 + v^2_{Q'_y} = v^2_{\max} = \left(\frac{v}{2}\right)^2. \tag{4}$$

This sets the speeds of all points, relative to a coordinate system whose velocity is $(v/2, 0, 0)$, equal to $|v_{\max}|$. Note that $v_{Q'_y}$ in Eq. (4) is relative to $v_y = 0$, not relative to

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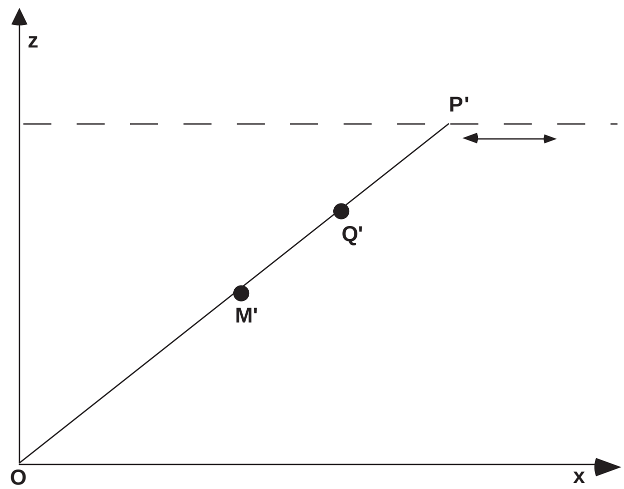


Fig. 1. Bar $OP'$ in the image plane $Oxz$. Point $O$ is fixed while $P'$ is translating horizontally back and forth. $M'$ is the midpoint (the perceptual center) and $Q'$ an arbitrary point on the bar.

$v_{M'_y}$. In the latter case, $v_{Q'_y}$ should be replaced by $v^*_{Q'_y} = v_{Q'_y} - v_{M'_y}$. Beghi et al. (1997) then obtain (second equation after Eq. (4), p. 13):

$$v_{Q'_y} = -v\sqrt{\lambda[1-\lambda]}. \tag{5}$$

Our first critique examines Eq. (5).[1] Our second critique examines Eq. (4).

## 2. The need to consider directions of a vector field

From Eq. (5), we note that

$$v_{O_y} = 0 \text{ and } v_{P'_y} = 0, \quad \text{when} \quad \lambda = 0 \text{ and } 1, \text{ respectively} \tag{6}$$

and (third equation after Eq. (4) of Beghi et al. (1997, p. 13))

$$v_{M'_y} = -\frac{v}{2} = -|v_{Q'_y\max}|, \quad \text{when } \lambda = \tfrac{1}{2}. \tag{7}$$

This means that while points $O$ and $P'$ remain on the image plane, point $M'$ moves out of the plane faster than any other point on the bar. There is no rigid motion compatible with this velocity profile. This velocity profile is also not compatible with the perceptual result reported in Beghi et al. (1997).

Beghi et al. (1997, p. 18) said that "in fact the velocities distribution, that we have obtained, is not exactly that of a rigid object, although it can be very closely approximated to one". However, this close approximation was achieved only for the distribution of the speeds, or absolute values, of the velocity vector field, as shown in Fig. 4 of Beghi et al. (1997). When the vector directions of the velocity field are taken into consideration, the distribution they have obtained is very different from that of a perceptually observed rigid bar. Fig. 2 is a modified version of the original Fig. 4 in Beghi et al. (1997), when the direction of motion is taken into consideration. Indeed, since $v_{Q'_y}$ in Eq. (5) is independent of time, it will predict that the bar will keep deforming indefinitely.

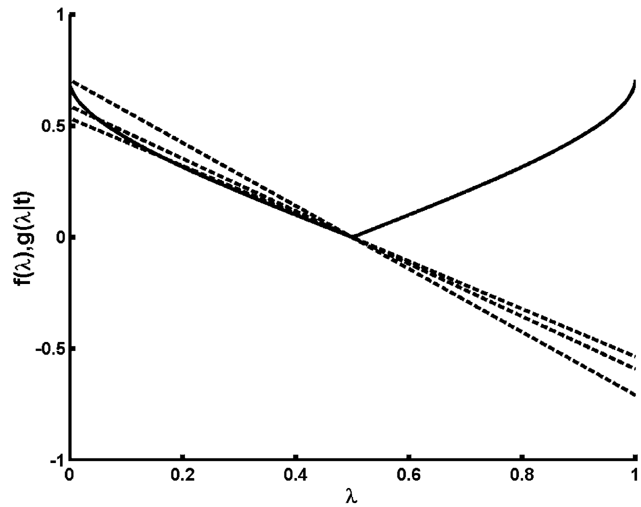


Fig. 2. A revision of Fig. 4 from Beghi et al. (1997). Here, motion directions in depth $y$ are taken into consideration. The curved, solid V-shape represents the motion profile $f(\lambda)$ derived from Beghi et al.'s minimal relative motion principle. The three straight dashed lines represent motion of a rigidly rotating bar $g(\lambda|t)$ at $t = 0, T/2$, and $T$, respectively.

## 3. The inadequate reference frame $(v/2, 0, 0)$

Recall that when deriving Eq. (4), the speeds of points along the bar are set to a constant $v/2$. These speeds are measured in a coordinate system that translates relative to the world coordinate system (the computer screen) with a velocity $(v/2, 0, 0)$. It is important to note that, since only the speed (or the magnitude) of a velocity vector is considered, but not the direction, Eq. (4) holds only relative to this coordinate system $(v/2, 0, 0)$. In other words, in a new coordinate system, the speeds are no longer equal. This is a problem, since $(v/2, 0, 0)$ is not the velocity of the perceptual center (the perceptual center has a velocity component in depth $v_{M'_y} \neq 0$).

Recall that the principle used by Beghi et al. (1997), as we quoted earlier, is to minimize the differences between the lengths of the velocity vectors of all points on the bar, with respect to the velocity of the center. Although

[1] Eq. (5) of Beghi et al. (1997) has a typo. The equation should be $v^*_{Q'} = \cdots. = v\sqrt{\frac{1}{2} - \lambda[1-\lambda]}$.

no exact equation is written in Beghi et al. (1997), to make all differences zero, this principle can be written as

$$v_{Q'_x}^2 + v_{Q'_y}^2 = v_{M'_x}^2 + v_{M'_y}^2, \tag{8}$$

where all velocity components are defined relative to the world coordinate system.[2] Technically, this minimization is solvable.

Instead, Beghi et al. (1997) have solved a different problem, in Eq. (4). These two problems are not equivalent. The solution to the latter (in Eq. (4)) is problematic, as we have shown in Section 2. More generally, this minimal relative motion principle appears to be problematic not only in its application to the case of the oscillating tilted bar discussed in this paper, but in its applications elsewhere also, as we have pointed out in Liu (2003) and Rokers and Liu (2004).

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[2] There is also an alternative interpretation. We can try to make $(v^*_{Q'_x})^2 + (v^*_{Q'_y})^2$ to be a constant, where $v^*_{Q'_x} = v_{Q'_x} - v_{M'_x}, v^*_{Q'_y} = v_{Q'_y} - v_{M'_y}$. This does not lead to reasonable results.